

Forced Vibration of an Aero-elastic Plate

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INTRODUCTION

In this paper we establish the existence of periodic solutions of a model equation which describes the motion of an aero-elastic plate with periodic forcing:

$$m \frac{\partial^2 u}{\partial t^2} + D \Delta^2 u - [u, F] + \tilde{\alpha} \frac{\partial u}{\partial t} + \tilde{\beta} \frac{\partial u}{\partial x} = \tilde{f}, \quad (0.1)$$

$$2\Delta^2 F + Eh[u, u] = 0 \quad \text{for } (t, x, y) \in R \times \Omega, \quad (0.2)$$

$$u = \frac{\partial u}{\partial n} = 0, \quad F = \frac{\partial F}{\partial n} = 0 \quad \text{for } (t, x, y) \in R \times \partial\Omega. \quad (0.3)$$

Here $\Delta^2 = (\partial^2/\partial x^2 + \partial^2/\partial y^2)^2$, the bracket $[,]$ is defined by

$$[p, q] = \frac{\partial^2 p}{\partial x^2} \frac{\partial^2 q}{\partial y^2} + \frac{\partial^2 p}{\partial y^2} \frac{\partial^2 q}{\partial x^2} - 2 \frac{\partial^2 p}{\partial x \partial y} \frac{\partial^2 q}{\partial x \partial y}, \quad (0.4)$$

$u(t, x, y)$ stands for the deflection of the plate from its flat position, $F(t, x, y)$ denotes the Airy stress function, $\tilde{f}(t, x, y)$ is a given external force and $m, D, \tilde{\alpha}, \tilde{\beta}, E$, and h are given physical constants. We suppose that the plate is homogeneous and is represented by Ω , a bounded open subset of R^2 with a smooth boundary. We denote by h and m the plate thickness and the plate mass per unit area, respectively. When the air is flowing at high supersonic speed in the x direction over the plate Ω , the plate motion gives rise to the aerodynamic pressure Δp which can be given by

$$\Delta p = \rho_{\infty} a_{\infty} \frac{\partial u}{\partial t} + \rho_{\infty} a_{\infty} U_{\infty} \frac{\partial u}{\partial x}, \quad (0.5)$$

where ρ_{∞} , a_{∞} , and U_{∞} denote the air density, the speed of sound, and the

air velocity, respectively, for the mean flow outside the boundary layer. Equation (0.5) is the simplest expression for Δp based on "piston theory" (see Dowell [3]). It is well known that (0.5) is a good approximation to the relation between aerodynamic pressure and plate motion provided the Mach number $\gg 1$ (see Dowell [3]). Since Δp accounts for the term $\tilde{\alpha} \partial u / \partial t + \tilde{\beta} \partial u / \partial x$, we expect $\tilde{\beta} / \tilde{\alpha}$ or U_∞ to be a large number. The term $\tilde{f}(t, x, y)$ is the external aerodynamic pressure independent of plate motion, e.g., turbulent pressure fluctuations. The constants D and E denote the plate stiffness and Young's modulus, respectively. The boundary condition (0.3) implies that the plate is clamped on the boundary and that the in-plane stress resultants vanish on the boundary. For the derivation of (0.1) to (0.5), the readers is referred to Dowell [3].

Morozov [6] discussed the same equation with $\tilde{\beta} = 0$. He established the existence of periodic solutions using the Galerkin approximation method: the original equation is approximated by finite dimensional systems of ordinary differential equations. For the equation in [6], each approximating system of ordinary differential equations is dissipative (for the definition, see Pliss [8]). Therefore, periodic solutions of approximating systems can be easily obtained via the theory of dissipative differential system. Similar problems were studied by Dmitrieva [2], Morozov [7], and Solop [11], using the same method as in [6]. Since $|\tilde{\beta} / \tilde{\alpha}|$ is allowed to be a large number, our system (0.1), (0.2) is not dissipative in general and we cannot apply the argument in [2], [6], [7], and [11]. Instead of the Galerkin method, we employ the method of regularization, which is particularly useful when a hyperbolic equation is discussed with a view to periodic solutions (see Strauss [12] and Lions [4]). The essence of this method is to obtain energy estimates (in the appropriate function space) independent of the coefficient of the artificial regularizing term. In the process of deriving energy estimates, $\tilde{\beta} \partial u / \partial x$ is an unwelcome term. Moreover, $\tilde{\beta} \partial u / \partial x$ cannot be controlled by means of $D \Delta^2 u$ if $|\tilde{\beta} / \tilde{\alpha}|$ is sufficiently large. Our method exploits the nonlinear term $[u, F]$ essentially to obtain energy estimates. Indeed, our whole scheme fails without this term. Rabinowitz [9] also used this term to obtain a certain estimate in the case of a stationary problem. From both mathematical and physical viewpoints, it seems interesting that the nonlinear term $[u, F]$ plays a crucial role in establishing the existence of periodic motion. Our main result is Theorem 2.1.

1. NOTATIONS AND MATHEMATICAL PRELIMINARIES

For the derivatives of a function $f(t, x, y)$, we use the following symbols and their combinations:

$$\begin{aligned}\partial_t f &= f_t = \frac{\partial f}{\partial t}, & \partial_x f &= \partial_x f = f_x = \frac{\partial f}{\partial x}, & \partial_y f &= \partial_y f = f_y = \frac{\partial f}{\partial y}, \\ \partial_{tt} f &= f_{tt} = \frac{\partial^2 f}{\partial t^2}, & \partial_{xx} f &= f_{xx} = \left(\frac{\partial}{\partial x}\right)^2 f, & \partial_{yy} f &= f_{yy} = \left(\frac{\partial}{\partial y}\right)^2 f,\end{aligned}$$

$\partial_t^m f = (\partial/\partial t)^m f$ for positive integers m . When E is a Banach space, $L^p((0, T); E)$, $1 \leq p \leq \infty$, denotes the set of all E -valued L^p functions with the norm $\|f\|_{L^p((0, T); E)} = (\int_0^T \|f(t)\|_E^p dt)^{1/p}$ for $1 \leq p < \infty$ and $\|f\|_{L^\infty((0, T); E)} = \text{ess sup}_{t \in (0, T)} \|f(t)\|_E$. $L_T^p(R; E)$, $1 \leq p \leq \infty$, is the set of all E -valued locally L^p functions defined on $(-\infty, \infty)$ with a period T , which is a Banach space with the norm $\|f\|_{L_T^p(R; E)} = \|f\|_{L^p((0, T); E)} \cdot C_T(R; E)$ is the set of all E -valued continuous functions defined on $(-\infty, \infty)$ with a period T , which is a Banach space with the norm $\|f\|_{C_T(R; E)} = \sup_{t \in [0, T]} \|f(t)\|_E$. Throughout this paper, Ω is a bounded open subset of R^2 with a C^∞ -boundary. $\partial/\partial n$ stands for the outward normal derivative on $\partial\Omega$. For a real number s , $H_0^s(\Omega)$ and $H^s(\Omega)$ are the Sobolev spaces as defined in Lions and Magenes [5]. When Q is an open subset of R^2 or R^3 , $\mathcal{D}^*(Q)$ stands for the space of distributions on Q .

We list several known facts which will be used later on.

LEMMA 1.1. *The mapping G defined by*

$$G: v \rightarrow w,$$

where w is a unique solution of

$$\begin{aligned}\Delta^2 w &= v && \text{in } \Omega, \\ w &= \frac{\partial w}{\partial n} = 0 && \text{on } \partial\Omega,\end{aligned}\tag{1.1}$$

is a continuous mapping from $H^s(\Omega)$ onto $H_0^2(\Omega) \cap H^{s+4}(\Omega)$, for $s \geq -2$, $s \neq -\frac{3}{2}, -\frac{1}{2}$ (see Lions and Magenes [5]).

LEMMA 1.2. *If $f \in H_0^2(0)$, then*

$$\|f_{xx}\|_{L^2(\Omega)} \leq \|Af\|_{L^2(\Omega)},\tag{1.2}$$

and if $f \in H_0^1(\Omega) \cap H^2(\Omega)$, then

$$\|f\|_{H^2(\Omega)} \leq C(\Omega) \|Af\|_{L^2(\Omega)},\tag{1.3}$$

where $C(\Omega)$ is a positive constant depending only on Ω .

Consider the eigenvalue problem:

$$\begin{aligned} \Delta^2 \varnothing_k &= \lambda_k \varnothing_k & \text{in } \Omega \\ \varnothing_k &= \frac{\partial \varnothing_k}{\partial n} = 0 & \text{on } \partial\Omega \end{aligned} \quad (1.4)$$

Then, we have

LEMMA 1.3. (i) *there are countably many eigenvalues $\{\lambda_k\}_{k=1}^\infty$ such that $0 < \lambda_1 \leq \lambda_2 \leq \dots$ and $\lambda_k \rightarrow \infty$ as $k \rightarrow \infty$.*

(ii) *the set of all eigenfunctions $\{\varnothing_k\}_{k=1}^\infty$ can be normalized such that*

$$\int_{\Omega} \varnothing_j \varnothing_k \, d\Omega = \delta_{jk}, \quad (1.5)$$

and it forms a complete basis for $L^2(\Omega)$.

(iii) $\Delta^{2m} \varnothing_k \in H_0^2(\Omega) \cap H^4(\Omega)$ for all integers $m \geq 0$ and $k \geq 1$.

LEMMA 1.4. *It holds that*

(i) $L^1(\Omega)$ is continuously embedded into $H^{-1-\varepsilon}(\Omega)$ for any $\varepsilon > 0$.

(ii) $H^s(\Omega)$ is continuously embedded into $C(\bar{\Omega})$ for $s > 1$.

(iii) $H^{1/2}(\Omega)$ is continuously embedded into $L^4(\Omega)$.

LEMMA 1.5. $(f, g) \rightarrow [f, g]$ is a continuous mapping

(i) from $H^2(\Omega) \times H^2(\Omega)$ into $L^1(\Omega)$,

(ii) from $H^s(\Omega) \times H^s(\Omega)$ into $L^2(\Omega)$, for $s \geq \frac{5}{2}$.

LEMMA 1.6. For all $f, g \in H_0^2(\Omega)$ and $h \in H^2(\Omega)$, $\int_{\Omega} [f, g] h \, d\Omega = \int_{\Omega} [f, h] g \, d\Omega$ holds.

The following lemma is indispensable for our energy estimates. The most elegant proof can be found in Ciarlet and Rabier [1].

LEMMA 1.7. If $f \in H_0^2(\Omega)$ and $[f, f] = 0$, a.e., in Ω , then $f = 0$ a.e., in Ω .

2. EXISTENCE OF PERIODIC SOLUTIONS

With the aid of Lemma 1.1, we write (0.1)–(0.3) as

$$u_{tt} + C_1 \Delta^2 u + C_2 [u, G[u, u]] + \alpha u_t + \beta u_x = f \quad \text{for } (t, x, y) \in R \times \Omega, \quad (2.1)$$

$$u = \frac{\partial u}{\partial n} = 0 \quad \text{for } (t, x, y) \in R \times \partial\Omega, \quad (2.2)$$

where $C_1 = D/m > 0$, $C_2 = Eh/2m > 0$, $\alpha = \tilde{\alpha}/m \neq 0$, $\beta = \tilde{\beta}/m$, and $f = \tilde{f}/m$. Our result is

THEOREM 2.1. *Suppose $f \in L_T^2(R; L^2(\Omega))$. Then, there is a solution to (2.1), (2.2) satisfying $u \in L_T^\infty(R; H_0^2(\Omega))$, $u_t \in L_T^\infty(R; L^2(\Omega))$.*

Before we prove this theorem, we have to take some preliminary steps. First we construct function spaces using Lemma 1.3:

$$\begin{aligned} \mathcal{X}_0 &= \left\{ \sum_{j=-\infty}^{\infty} \sum_{k=1}^{\infty} a_{jk} \exp\left(i \frac{2\pi}{T} jt\right) \varnothing_k : \overline{a_{j,k}} \right. \\ &\quad \left. = a_{-j,k}, \sum_{j=-\infty}^{\infty} \sum_{k=1}^{\infty} |a_{jk}|^2 < \infty \right\} \end{aligned} \quad (2.3)$$

$$\begin{aligned} \mathcal{X}_1 &= \left\{ \sum_{j=-\infty}^{\infty} \sum_{k=1}^{\infty} a_{jk} \exp\left(i \frac{2\pi}{T} jt\right) \varnothing_k : \overline{a_{j,k}} \right. \\ &\quad \left. = a_{-j,k}, \sum_{j=-\infty}^{\infty} \sum_{k=1}^{\infty} (j^4 + \lambda_k^2 + j^2 \lambda_k^2) |a_{jk}|^2 < \infty \right\}. \end{aligned} \quad (2.4)$$

Then \mathcal{X}_0 and \mathcal{X}_1 are Hilbert spaces equipped with obvious inner products. Next, we consider the equation:

$$\begin{aligned} \eta(-\partial_t^6 u + \Delta^6 u) + \varepsilon \Delta^2 u_t + u_{tt} + C_1 \Delta^2 u \\ + C_2 [u, G[u, u]] + \alpha u_t + \beta u_x = f, \end{aligned} \quad (2.5)$$

where η, ε are given positive constants and we assume $\alpha > 0$ (this assumption will be dropped later on). We will obtain a solution of (2.5) as a fixed point of a certain operator. Suppose $v \in \mathcal{X}_1$. Then, $v_{tt} \in L_T^2(R; L^2(\Omega))$, $v_t \in L_T^2(R; H_0^2(\Omega) \cap H^4(\Omega))$ and $v \in L_T^2(R; H_0^2(\Omega) \cap H^4(\Omega))$, which imply that $v \in C_T(R; H_0^2(\Omega) \cap H^4(\Omega))$ after possibly a modification on a set of measure zero in R . Therefore, $v \rightarrow [v, G[v, v]]$ is a continuous

mapping from \mathcal{X}_1 into $C_T(R; L^2(\Omega))$ by Lemmas 1.1 and 1.5. Since $\mathcal{X}_0 = L_T^2(R; L^2(\Omega))$, the mapping Γ_ε defined by

$$v \rightarrow -\{\varepsilon \Delta^2 v_t + v_{tt} + C_1 \Delta^2 v + C_2[v, G[v, v]] + \alpha v_t + \beta v_x\} + f \quad (2.6)$$

is a continuous mapping from \mathcal{X}_1 into \mathcal{X}_0 . Next we define a mapping $\mathcal{F}_{\varepsilon, \eta}$ by

$$v \rightarrow \sum_{j=-\infty}^{\infty} \sum_{k=1}^{\infty} \frac{1}{\eta} \frac{1}{(2\pi j/T)^6 + \lambda_k^3} b_{jk} \exp\left(i \frac{2\pi}{T} jt\right) \varnothing_k, \quad (2.7)$$

where

$$b_{jk} = \frac{1}{T} \int_0^T \int_{\Omega} (\Gamma_\varepsilon v) \varnothing_k \exp\left(-i \frac{2\pi}{T} jt\right) d\Omega dt.$$

Then it is obvious that $\mathcal{F}_{\varepsilon, \eta}$ is a continuous compact mapping from \mathcal{X}_1 into itself and that its fixed point satisfies (2.5) in $L_T^2(R; L^2(\Omega))$.

LEMMA 2.2. *For each $\varepsilon, \eta > 0$, the equation $u = \mathcal{F}_{\varepsilon, \eta} u$ has a solution $u_{\varepsilon, \eta}$ in \mathcal{X}_1 satisfying*

$$\|u_{\varepsilon, \eta}\|_{L_T^2(R; H_0^2(\Omega))} \leq M, \quad (2.8)$$

$$\|\partial_t u_{\varepsilon, \eta}\|_{L_T^2(R; L^2(\Omega))} \leq M, \quad (2.9)$$

$$\sqrt{\varepsilon} \|\partial_t u_{\varepsilon, \eta}\|_{L_T^2(R; H_0^2(\Omega))} \leq M, \quad (2.10)$$

where M is a positive constant independent of $\varepsilon, \eta > 0$.

The proof of this lemma depends on the following version of the Leray–Schauder fixed point theorem.

LEMMA 2.3. (Schaefer [10]). *Let \mathcal{F} be a continuous compact mapping of a Banach space \mathcal{X} into itself, and suppose there is a constant M such that $\|z\| < M$ for all $z \in \mathcal{X}$ and $\lambda \in [0, 1]$ satisfying $z = \lambda \mathcal{F} z$. Then, \mathcal{F} has a fixed point in \mathcal{X} .*

We also need the following lemma to obtain the essential energy estimate.

LEMMA 2.4. *Denote by $S_{K, L}$ the set of all functions u satisfying $u \in L_T^2(R; H_0^2(\Omega))$, $u_t \in L_T^2(R; L^2(\Omega))$, $\Delta G[u, u] \in L_T^2(R; L^2(\Omega))$, and*

$$\begin{aligned} & \int_0^T \int_{\Omega} (\Delta G[u, u])^2 d\Omega dt + \int_0^T \int_{\Omega} (\Delta u)^2 d\Omega dt + \int_0^T \int_{\Omega} (u_t)^2 d\Omega dt \\ & \leq K \int_0^T \int_{\Omega} u^2 d\Omega dt + L, \end{aligned} \quad (2.11)$$

where K and L are given positive constants. Then, there is a positive constant $M_{K,L}$ such that

$$\int_0^T \int_{\Omega} u^2 d\Omega dt \leq M_{K,L} \quad \text{for all } u \in S_{K,L}. \quad (2.12)$$

Proof. The idea of proof is similar to that of Lemma 2.1 in Rabinowitz [9]. Suppose there is no such $M_{K,L}$. Then, there is a sequence $\{u_n\}_{n=1}^{\infty}$ in $S_{K,L}$ such that $\xi_n = \text{def } \|u_n\|_{L_T^2(R; L^2(\Omega))}$ tends to infinity as $n \rightarrow \infty$. Define $v_n = u_n/\xi_n$. Then we have

$$\begin{aligned} & \xi_n^2 \int_0^T \int_{\Omega} (\Delta G[v_n, v_n])^2 d\Omega dt + \int_0^T \int_{\Omega} (\Delta v_n)^2 d\Omega dt \\ & + \int_0^T \int_{\Omega} (\partial_t v_n)^2 d\Omega dt \leq K + \frac{L}{\xi_n^2} \quad \text{for all } n. \end{aligned} \quad (2.13)$$

By virtue of (2.13), we observe that there is a function $v \in L_T^2(R; H_0^2(\Omega))$ and a subsequence still denoted by $\{v_n\}$ such that $v_n \rightarrow v$ strongly in $L_T^2(R; L^2(\Omega))$, $\partial_i \partial_j v_n \rightarrow \partial_i \partial_j v$ weakly in $L_T^2(R; L^2(\Omega))$, $i, j = 1, 2$, and $\partial_t v_n \rightarrow \partial_t v$ weakly in $L_T^2(R; L^2(\Omega))$. Hence for each $\varnothing \in C^\infty([0, T]; H^4(\Omega))$, $\int_0^T \int_{\Omega} [v_n, \varnothing] v_n d\Omega dt$ converges to $\int_0^T \int_{\Omega} [v, \varnothing] v d\Omega dt$, and by Lemma 1.6, it holds that

$$\int_0^T \int_{\Omega} [v_n, v_n] \varnothing d\Omega dt = \int_0^T \int_{\Omega} [v_n, \varnothing] v_n d\Omega dt \quad (2.14)$$

and

$$\int_0^T \int_{\Omega} [v, v] \varnothing d\Omega dt = \int_0^T \int_{\Omega} [v, \varnothing] v d\Omega dt. \quad (2.15)$$

Consequently, for any $\Psi \in C_0^\infty(R \times \Omega)$ and any integer k , $\int_{kT}^{(k+1)T} \int_{\Omega} [v_n, v_n] G\Psi d\Omega dt$ converges to $\int_{kT}^{(k+1)T} \int_{\Omega} [v, v] G\Psi d\Omega dt$ from which it follows that $\Delta G[v_n, v_n]$ converges to $\Delta G[v, v]$ in $\mathcal{D}'(R \times \Omega)$ and hence, $\Delta G[v, v] \in L_T^2(R; L^2(\Omega))$ and by (2.13),

$$\begin{aligned} & \|\Delta G[v, v]\|_{L_T^2(R; L^2(\Omega))} \\ & \leq \varliminf_{n \rightarrow \infty} \|\Delta G[v_n, v_n]\|_{L_T^2(R; L^2(\Omega))} = 0. \end{aligned} \quad (2.16)$$

This implies $v = 0$, a.e., in $(0, T) \times \Omega$ by Lemma 1.7, which contradicts the fact that $\|v\|_{L_T^2(R; L^2(\Omega))} = 1$ since $\|v_n\|_{L_T^2(R; L^2(\Omega))} = 1$ for all n .

Remark 2.5. We have used Lemma 1.7 essentially, which was not used in Rabinowitz [9].

Proof of Lemma 2.2. Fix any given $\eta > 0$, and $\varepsilon > 0$, omit the subscripts η, ε . Suppose u is a solution of $u = \lambda \mathcal{F}u$ in \mathcal{X}_1 for some $\lambda \in [0, 1]$. Then, recalling (2.7), we notice that $\partial_t^6 u, \Delta^6 u \in L_T^2(R; L^2(\Omega)), \Delta^{2m} u \in L_T^2(R; H_0^2(\Omega) \cap H^4(\Omega)), m = 0, 1, 2$, and that

$$\eta(-\partial_t^6 u + \Delta^6 u) + \lambda(\varepsilon \Delta^2 u_t + u_{tt} + C_1 \Delta^2 u + C_2[u, G[u, u]]) + \alpha u_t + \beta u_x = \lambda f \quad (2.17)$$

holds in $L_T^2(R; L^2(\Omega))$. Multiplying both sides of (2.17) by u, u_t and integrating over $(0, T) \times \Omega$, we obtain, respectively,

$$\begin{aligned} \eta \int_0^T \int_{\Omega} \{(\partial_t^3 u)^2 + (\Delta^3 u)^2\} d\Omega dt + \lambda \int_0^T \int_{\Omega} \{-u_t^2 + C_1(\Delta u)^2 \\ C_2(\Delta G[u, u])^2\} d\Omega dt = \lambda \int_0^T \int_{\Omega} fu d\Omega dt \end{aligned} \quad (2.18)$$

and

$$\lambda \int_0^T \int_{\Omega} \{\varepsilon(\Delta u_t)^2 + \alpha u_t^2 + \beta u_t u_x\} d\Omega dt = \lambda \int_0^T \int_{\Omega} fu_t d\Omega dt. \quad (2.19)$$

Since $u \equiv 0$ when $\lambda = 0$, we proceed by assuming that $\lambda > 0$. From (2.18) (2.19), we derive that

$$\begin{aligned} \int_0^T \int_{\Omega} \{-u_t^2 + C_1(\Delta u)^2 + C_2(\Delta G[u, u])^2\} d\Omega dt \\ \leq \frac{1}{2} \int_0^T \int_{\Omega} f^2 d\Omega dt + \frac{1}{2} \int_0^T \int_{\Omega} u^2 d\Omega dt \end{aligned} \quad (2.20)$$

and

$$\int_0^T \int_{\Omega} u_t^2 d\Omega dt \leq 2 \int_0^T \int_{\Omega} \left(\frac{\beta}{\alpha}\right)^2 u_x^2 d\Omega dt + \frac{2}{\alpha^2} \int_0^T \int_{\Omega} f^2 d\Omega dt. \quad (2.21)$$

Combining (2.20), (2.21), and the inequality

$$\begin{aligned} 2 \left(\frac{\beta}{\alpha}\right)^2 \int_0^T \int_{\Omega} u_x^2 d\Omega dt = 2 \left(\frac{\beta}{\alpha}\right)^2 \int_0^T \left| \int_{\Omega} uu_{xx} d\Omega \right| dt \\ \leq \frac{C_1}{4} \int_0^T \int_{\Omega} (\Delta u)^2 d\Omega dt + \frac{4}{C_1} \left(\frac{\beta}{\alpha}\right)^4 \int_0^T \int_{\Omega} u^2 d\Omega dt \end{aligned} \quad (2.22)$$

where we have used Lemma 1.2, we obtain

$$\begin{aligned} & \int_0^T \int_{\Omega} \left\{ u_t^2 + \frac{C_1}{2} (\Delta u)^2 + C_2 (\Delta G[u, u])^2 \right\} d\Omega dt \\ & \leq \left(\frac{4}{\alpha^2} + \frac{1}{2} \right) \int_0^T \int_{\Omega} f^2 d\Omega dt + \left\{ \frac{1}{2} + \frac{8}{C_1} \left(\frac{\beta}{\alpha} \right)^4 \right\} \int_0^T \int_{\Omega} u^2 d\Omega dt. \end{aligned} \quad (2.23)$$

By means of Lemma 2.4, we see that there is a constant $M > 0$ independent of $\varepsilon, \eta, \lambda$ such that

$$\int_0^T \int_{\Omega} u^2 d\Omega dt \leq M, \quad (2.24)$$

$$\int_0^T \int_{\Omega} \{ u_t^2 + (\Delta u)^2 + (\Delta G[u, u])^2 \} d\Omega dt \leq M. \quad (2.25)$$

Now (2.18) yields

$$\int_0^T \int_{\Omega} \{ (\partial_t^3 u)^2 + (\Delta^3 u)^2 \} d\Omega dt \leq \frac{1}{\eta} M \quad \text{for all } \lambda \in [0, 1], \quad (2.26)$$

where M is a positive constant independent of λ, ε , and η . Consequently, we arrive at

$$\|u\|_{\mathcal{X}_1} < \text{constant independent of } \lambda \in [0, 1]. \quad (2.27)$$

With the help of Lemma 2.3, (2.19), (2.24), and (2.25), we conclude that for each $\varepsilon, \eta > 0$, there is a solution $u_{\varepsilon, \eta} \in \mathcal{X}_1$ of the equation

$$u_{\varepsilon, \eta} = \mathcal{F}_{\varepsilon, \eta} u_{\varepsilon, \eta} \quad (2.28)$$

satisfying the estimate

$$\begin{aligned} & \int_0^T \int_{\Omega} \{ u_{\varepsilon, \eta}^2 + \varepsilon (\Delta \partial_t u_{\varepsilon, \eta})^2 + (\partial_t u_{\varepsilon, \eta})^2 + (\Delta u_{\varepsilon, \eta})^2 \\ & \quad + (\Delta G[u_{\varepsilon, \eta}, u_{\varepsilon, \eta}])^2 \} d\Omega dt \leq M, \end{aligned} \quad (2.29)$$

where M is a positive constant independent of $\varepsilon, \eta > 0$. Now the proof of Lemma 2.2 is complete.

Next we consider the equation

$$\varepsilon \Delta^2 u_t + u_{tt} + C_1 \Delta^2 u + C_2 [u, G[u, u]] + \alpha u_t + \beta u_x = f \quad (2.30)$$

for which our assertion is

LEMMA 2.6. For each $\varepsilon > 0$, (2.30) has a solution u_ε satisfying $u_\varepsilon \in C_T(R; H_0^2(\Omega))$, $\partial_t u_\varepsilon \in L_T^\infty(R; L^2(\Omega)) \cap L_T^2(R; H_0^2(\Omega))$, and

$$\|u_\varepsilon\|_{L_T^\infty(R; H_0^2(\Omega))} \leq M, \quad (2.31)$$

$$\|\partial_t u_\varepsilon\|_{L_T^\infty(R; L^2(\Omega))} \leq M, \quad (2.32)$$

where M is a positive constant independent of $\varepsilon > 0$.

Proof of Lemma 2.6. As above, let $u_{\varepsilon, \eta}$ be a solution of (2.28) satisfying (2.29). Fix any $\varepsilon > 0$. Then, by (2.29), we can extract a Cauchy sequence $\{u_{\varepsilon, \eta_k}\}_{k=1}^\infty$ in \mathcal{X}_0 from the set $\{u_{\varepsilon, \eta}\}_{\eta > 0}$ such that $\eta_k \rightarrow 0$ as $k \rightarrow \infty$. Set $v_k = u_{\varepsilon, \eta_k}$. Then each $v_k \in \mathcal{X}_1$ and there is a function v in \mathcal{X}_0 such that $v_k \rightarrow v$ strongly in \mathcal{X}_0 , $\partial_i \partial_j \partial_t v_k \rightarrow \partial_i \partial_j \partial_t v$ weakly in \mathcal{X}_0 , $i, j = 1, 2$, $\partial_t v_k \rightarrow \partial_t v$ weakly in \mathcal{X}_0 , $\partial_i \partial_j v_k \rightarrow \partial_i \partial_j v$ weakly in \mathcal{X}_0 , and hence, $v, \partial_t v \in L_T^2(R; H_0^2(\Omega))$. It follows that $v \in C_T(R; H_0^2(\Omega))$ and $[v, G[v, v]] \in C_T(R; L^1(\Omega))$ after possibly a modification on a set of measure zero in R . Using the same argument as in the proof of Lemma 2.4, it can be easily shown that $G[v_k, v_k]$ converges to $G[v, v]$ in $\mathcal{D}^*(R \times \Omega)$ and hence, $\partial_i \partial_j G[v_k, v_k] \rightarrow \partial_i \partial_j G[v, v]$ weakly in \mathcal{X}_0 , $i, j = 1, 2$. Now for each k , v_k satisfies (2.5) so that

$$\begin{aligned} & \int_{-\infty}^{\infty} \int_{\Omega} \eta_k v_k (-\partial_t^6 \varnothing + \Delta^6 \varnothing) d\Omega dt \\ & + \int_{-\infty}^{\infty} \int_{\Omega} v_k (-\varepsilon \Delta^2 \varnothing_t + \varnothing_{tt} + C_1 \Delta^2 \varnothing - \alpha \varnothing_t - \beta \varnothing_x) d\Omega dt \\ & + \int_{-\infty}^{\infty} \int_{\Omega} C_2 [v_k, G[v_k, v_k]] \varnothing d\Omega dt = \int_{-\infty}^{\infty} \int_{\Omega} f \varnothing d\Omega dt \end{aligned} \quad (2.33)$$

for all $\varnothing \in C_0^\infty(R \times \Omega)$. Next we observe that $\int_{-\infty}^{\infty} \int_{\Omega} [\varnothing, G[v_k, v_k]] v_k d\Omega dt$ converges to $\int_{-\infty}^{\infty} \int_{\Omega} [\varnothing, G[v, v]] v d\Omega dt$, for each $\varnothing \in C_0^\infty(R \times \Omega)$, and thus, by Lemma 1.6, $\int_{-\infty}^{\infty} \int_{\Omega} [v_k, G[v_k, v_k]] \varnothing d\Omega dt$ converges to $\int_{-\infty}^{\infty} \int_{\Omega} [v, G[v, v]] \varnothing d\Omega dt$. Therefore, by letting $k \rightarrow \infty$, it holds that

$$\begin{aligned} & \int_{-\infty}^{\infty} \int_{\Omega} v (-\varepsilon \Delta^2 \varnothing_t + \varnothing_{tt} + C_1 \Delta^2 \varnothing - \alpha \varnothing_t - \beta \varnothing_x) d\Omega dt \\ & + \int_{-\infty}^{\infty} \int_{\Omega} C_2 [v, G[v, v]] \varnothing d\Omega dt = \int_{-\infty}^{\infty} \int_{\Omega} f \varnothing d\Omega dt, \end{aligned} \quad (2.34)$$

for all $\varnothing \in C_0^\infty(R \times \Omega)$. So v is a solution of

$$\varepsilon \Delta^2 v_t + v_{tt} + C_1 \Delta^2 v + C_2 [v, G[v, v]] + \alpha v_t + \beta v_x = f, \quad (2.35)$$

satisfying $v \in C_T(R; H_0^2(\Omega))$, $\partial_t v \in L_T^2(R; H_0^2(\Omega))$, $[v, G[v, v]] \in C_T(R; L^1(\Omega))$ and

$$\int_0^T \int_{\Omega} \{v^2 + \varepsilon(\Delta v_t)^2 + v_t^2 + (\Delta v)^2 + (\Delta G[v, v])^2\} d\Omega dt \leq M, \quad (2.36)$$

where M is a positive constant independent of $\varepsilon > 0$. To proceed further let us set

$$w_v(t, x, y) = v * \rho_v = \int_{-\infty}^{\infty} v(t - \tau, x, y) \rho_v(\tau) d\tau, \quad (2.37)$$

where $\rho_v(\cdot)$ is the Friedrichs mollifier. Then, w_v satisfies

$$\begin{aligned} \varepsilon \Delta^2 \partial_t w_v + \partial_{tt} w_v + C_1 \Delta^2 w_v + C_2 [w_v, G[w_v, w_v]] \\ + \alpha \partial_t w_v + \beta \partial_x w_v = f_v + g_v \end{aligned} \quad (2.38)$$

in $C_T^\infty(R; H^{-2}(\Omega))$, where $f_v(t, x, y) = \int_{-\infty}^{\infty} f(t - \tau, x, y) \rho_v(\tau) d\tau$ and $g_v = C_2 [w_v, G[w_v, w_v]] - C_2 [v, G[v, v]] * \rho_v$. Since $v \in C_T(R; H_0^2(\Omega))$ and $[v, G[v, v]] \in C_T(R; L^1(\Omega))$, we see that

$$\lim_{v \rightarrow 0} \|G[v, v] - G[w_v, w_v]\|_{C_T(R; H_0^2(\Omega))} = 0, \quad (2.39)$$

$$\lim_{v \rightarrow 0} \|[v, G[v, v]] - [v, G[v, v]] * \rho_v\|_{C_T(R; L^1(\Omega))} = 0, \quad (2.40)$$

$$\lim_{v \rightarrow 0} \|[w_v, G[w_v, w_v]] - [v, G[v, v]]\|_{C_T(R; L^1(\Omega))} = 0, \quad (2.41)$$

where we have used Lemmas 1.1, 1.4, and 1.5. Let us multiply both sides of (2.38) by $\partial_t w_v$ and integrate over Ω to arrive at

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} \frac{1}{2} \{(\partial_t w_v)^2 + C_1 (\Delta w_v)^2 + \frac{1}{2} C_2 (\Delta G[w_v, w_v])^2\} d\Omega \\ + \int_{\Omega} \{\varepsilon (\Delta \partial_t w_v)^2 + \alpha (\partial_t w_v)^2 + \beta (\partial_t w_v)(\partial_x w_v)\} d\Omega \\ = \int_{\Omega} f_v \partial_t w_v d\Omega + \int_{\Omega} g_v \partial_t w_v d\Omega. \end{aligned} \quad (2.42)$$

As a consequence of (2.39)–(2.41), there is a positive number $\delta(\varepsilon, v)$ depending on ε, v such that for all $0 < v \leq \delta(\varepsilon, v)$,

$$\|G[w_v, w_v]\|_{L_T^2(R; H_0^2(\Omega))} \leq \|G[v, v]\|_{L_T^2(R; H_0^2(\Omega))} + 1 \quad (2.43)$$

and

$$\|g_v\|_{C_T(R; L^1(\Omega))} \leq \sqrt{\varepsilon}. \quad (2.44)$$

Next we define

$$E_v(t) = \frac{1}{2} \int_{\Omega} \{(\partial_t w_v)^2 + C_1(\Delta w_v)^2 + \frac{1}{2} C_2(\Delta G[w_v, w_v])^2\} d\Omega. \quad (2.45)$$

Then, (2.42), together with (2.43), (2.44), implies that

$$\frac{d}{dt} E_v(t) \leq K E_v(t) + \int_{\Omega} f_v^2 d\Omega + K \quad (2.46)$$

and

$$\frac{d}{dt} E_v(t) \geq -K E_v(t) - 2\varepsilon \int_{\Omega} (\Delta \partial_t w_v)^2 d\Omega - \int_{\Omega} f_v^2 d\Omega - K, \quad (2.47)$$

for all $0 < v \leq \delta(\varepsilon, v)$. Here K is a positive constant depending only on $C_1, \alpha, \beta, \Omega$, and we have used the inequalities:

$$\|\partial_x w_v\|_{L^2(\Omega)} \leq M(\Omega) \|\Delta w_v\|_{L^2(\Omega)} \quad \text{for each } t, \quad (2.48)$$

$$\begin{aligned} \left| \int_{\Omega} g_v \partial_t w_v d\Omega \right| &\leq \|\partial_t w_v\|_{L^\infty(\Omega)} \int_{\Omega} |g_v| d\Omega \\ &\leq M(\Omega) \|\Delta \partial_t w_v\|_{L^2(\Omega)} \|g_v\|_{L^1(\Omega)} \\ &\leq \frac{\varepsilon}{2} \|\Delta \partial_t w_v\|_{L^2(\Omega)}^2 + \frac{M(\Omega)^2}{2\varepsilon} \|g_v\|_{L^1(\Omega)}^2 \quad \text{for each } t, \end{aligned} \quad (2.49)$$

where $M(\Omega)$ denotes positive constants depending only on Ω . By virtue of (2.36) and (2.43), it is apparent that for all $0 < v \leq \delta(\varepsilon, v)$,

$$\int_0^T \int_{\Omega} \{w_v^2 + \varepsilon(\Delta \partial_t w_v)^2 + (\partial_t w_v)^2 + (\Delta w_v)^2 + (\Delta G[w_v, w_v])^2\} d\Omega dt \leq M, \quad (2.50)$$

where M is a positive constant independent of v, ε . Combining (2.46), (2.47), (2.50), and the inequality

$$\int_0^T \int_{\Omega} f_v^2 d\Omega dt \leq \int_0^T \int_{\Omega} f^2 d\Omega dt \quad \text{for all } v > 0, \quad (2.51)$$

we deduce that for each $0 < v \leq \delta(\varepsilon, v)$,

$$|E_v(t_1) \exp(Kt_1) - E_v(t_2) \exp(Kt_2)| \leq M \quad (2.52)$$

hold for all $t_1, t_2 \in [0, T]$, where M is a positive constant independent of v, ε . Finally (2.50) together with (2.52) yields

$$\sup_{t \in [0, T]} |E_v(t)| \leq M \quad \text{for all } 0 < v \leq \delta(\varepsilon, v), \quad (2.53)$$

where M is a positive constant independent of v, ε . Since $w_v \rightarrow v$ strongly in $C_T(R; H_0^2(\Omega))$ as $v \rightarrow 0$, we obtain, from (2.53),

$$\|v\|_{C_T(R; H_0^2(\Omega))} \leq M \quad (2.54)$$

and

$$\|v_t\|_{L_T^\infty(R; L^2(\Omega))} \leq M, \quad (2.55)$$

where M is a positive constant independent of $\varepsilon > 0$. This completes the proof of Lemma 2.6.

Now we are ready to give

Proof of Theorem 2.1. For each $\varepsilon > 0$, let u_ε be a solution of (2.30), satisfying (2.31), and (2.32). Then we can extract a sequence denoted again by $\{u_\varepsilon\}$ such that for some function u , $u_\varepsilon \rightarrow u$ weak* in $L_T^\infty(R; H_0^2(\Omega))$, $\partial_i u_\varepsilon \rightarrow \partial_i u$ weak* in $L_T^\infty(R; L^2(\Omega))$ as $\varepsilon \rightarrow 0$, and hence, $u_\varepsilon \rightarrow u$ strongly in $L_T^2(R; L^2(\Omega))$. From these, it can be easily shown that $\partial_i \partial_j G[u_\varepsilon, u_\varepsilon] \rightarrow \partial_i \partial_j G[u, u]$ weak* in $L_T^\infty(R; L^2(\Omega))$, $i, j = 1, 2$, by the same argument as in the proof of Lemma 2.4. Thus, $[u_\varepsilon, G[u_\varepsilon, u_\varepsilon]]$ converges to $[u, G[u, u]]$ in $\mathcal{D}^*(R \times \Omega)$ as $\varepsilon \rightarrow 0$, from which it follows that u is a solution of

$$u_{tt} + C_1 \Delta^2 u + C_2[u, G[u, u]] + \alpha u_t + \beta u_x = f \quad (2.56)$$

in $\mathcal{D}^*(R \times \Omega)$ such that

$$u \in L_T^\infty(R; H_0^2(\Omega)) \quad (2.57)$$

and

$$u_t \in L_T^\infty(R; L^2(\Omega)). \quad (2.58)$$

We have proved Theorem 2.1 under the assumption $\alpha > 0$. If $\alpha < 0$, we simply substitute $-\varepsilon \Delta^2 \partial_t u$ for $\varepsilon \Delta^2 \partial_t u$ in the above argument to obtain the same result.

Remark 2.7. The purpose of the term $\varepsilon \Delta^2 \partial_t u$ was to gain L^∞ -regularity in time variable.

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